5.2 \( N = \# \) of years until \( X_1 > X_2 \) for the first time. Using independence argument and the fact that \( F(x) \sim U(0,1) \),

\[
P(X_2 > X_1) = \int_0^{\infty} P(X_2 > x | X_1 = x) f(x) \, dx = \int_0^{\infty} P(X_2 > x | X_1 = x) f(x) \, dx = E(1 - F(x)) = 1/2
\]

\[
P(X_2 > X_1, X_2 \leq X_1) = \int_0^{\infty} P(X_2 > x, X_2 \leq X_1 | X_1 = x) f(x) \, dx = \int_0^{\infty} P(X_2 > x) P(X_2 \leq x | X_1 = x) f(x) \, dx = E(1 - F(x))F(x) = 1/2 - 1/3, \ldots
\]

\[
N = k : P(X_{k+1} > X_1, X_1 \leq X_k, 2 \leq i \leq k) = E((1 - F(x))(F(x)^{k-1}) = 1/k - 1/(k+1)
\]

5.3 Note that \( Y_i \sim \text{Bernoulli} \) with \( p_i = P[X_i \geq \mu] = 1 - F(\mu) \) for each \( i \). Since the \( Y_i \)'s are iid Bernoulli, \( \sum_{i=1}^{n} Y_i \sim \text{binomial}(n, p = 1 - F(\mu)) \).

5.8 Let \( \mu_i = E(X_i) \). Then

\[
\text{Var} \left( \sum_{i=1}^{n} X_i \right) = \text{Var}(X_1 + X_2 + \ldots + X_n)
\]

\[
= E[(X_1 + X_2 + \ldots + X_n)^2] - (\mu_1 + \mu_2 + \ldots + \mu_n)^2
\]

\[
= E[(X_1^2 + X_2^2 + \ldots + X_n^2)] - (\mu_1 + \mu_2 + \ldots + \mu_n)^2
\]

\[
= \sum_{i=1}^{n} E(X_i^2) + 2 \sum_{1 \leq i < j \leq n} E(X_i X_j) - (\sum_{1 \leq i \leq n} \mu_i)^2
\]

\[
= \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)
\]

5.18 From the CLT we have, approximately, \( X_{\bar{n}} \sim \text{N}(n \mu, \sigma^2/n) \), \( X_{\bar{2}} \sim \text{N}(n \mu, \sigma^2/n) \). Since \( X_1 \) and \( X_2 \) are independent, \( X_1 - X_2 \sim \text{N}(0, 2\sigma^2/n) \). Thus, we want

\[
0.99 \approx P(\left| X_{\bar{1}} - X_{\bar{2}} \right| < \sigma/5) = P \left( \frac{\sigma/5}{\sigma/\sqrt{5n/2}} < \frac{X_{\bar{1}} - X_{\bar{2}}}{\sigma/\sqrt{5n/2}} < -\frac{\sigma/5}{\sigma/\sqrt{5n/2}} \right) \approx P(\frac{1}{\sqrt{5/2}} < Z < \frac{1}{\sqrt{5/2}})
\]

where \( Z \sim \text{N}(0,1) \). Thus we need \( P(Z \geq \sqrt{5/2}) = .005 \). From Table 1, \( \sqrt{5/2} = 2.576 \), which implies \( n = 50(2.576)^2 \approx 332 \).

5.19 We know that \( \sigma^2 \bar{n} = 9/100 \). Use Chebychev's Inequality to get:

\[
P(-k/10 < X - \mu < k/10) \geq 1 - 1/k^2
\]

We need \( 1 - 1/k^2 \geq .90 \), which implies \( k \geq \sqrt{10} = 3.16 \) and \( k/10 \approx .9487 \). Thus \( P(-.9487 < X - \mu < .9487) \geq .90 \) by Chebychev's Inequality.

Using the CLT, \( \bar{X} \) is approximately \( \text{N}(\mu, \sigma^2/5) \) with \( \sigma^2/5 = 3.3 \). And \( (\bar{X} - \mu)/\sqrt{n} \sim \text{N}(0,1) \).

Thus

\[
0.9 \approx P \left( -1.645 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.645 \right) = P(-.4935 < X - \mu < .4935).
\]

Thus, we again see the conservativeness of Chebychev's Inequality, yielding bounds on \( \bar{X} - \mu \) that are almost twice as big as the normal approximation. Moreover, with a sample of size 100, \( \bar{X} \) is probably very close to normally distributed, even if the underlying \( X \) distribution is not close to normal.

5.22 (a)

\[
E(Y) = E(E(Y|N)) = E(2N) = 2B
\]

\[
Var(Y) = Var(E(Y|N)) + E(Var(Y|N)) = Var(2N) + E(4N) = 4\theta + 4\theta = 8\theta
\]

(b) Check that \( \lim_{n\to\infty} M_2(t) = \exp(t^2/2) \), where \( Z = (Y - EY)/\sqrt{\text{Var}(Y)} \).

\[
M(t) = E(e^{tY}) = E(e^{tN}) = E((1 + 1/(2t))^{2N}) = \prod_{n=0}^{\infty} (1 + 1/(2t))^{n} = e^{t - t/(1 - 2t)} = e^{t/(1 - 2t)}
\]

\[
M(t) = E(e^{tY}) = E(e^{t(2N/\sqrt{2\theta})}) = \exp \left( \frac{t}{\sqrt{2\theta}} - \frac{t}{\sqrt{2\theta}} \right) = \exp(t^2/2)
\]

\[
= \exp \left( \frac{t^2}{2} - \frac{t}{\sqrt{2\theta}} \right) = \exp(t^2/2)
\]